

## RAINBOW SETS IN THE INTERSECTION OF TWO MATROIDS

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ABSTRACT. Given sets  $F_1, \dots, F_n$ , a *partial rainbow function* is a partial choice function of the sets  $F_i$ . A *partial rainbow set* is the range of a partial rainbow function. Aharoni and Berger [2] conjectured that if  $\mathcal{M}$  and  $\mathcal{N}$  are matroids on the same ground set, and  $F_1, \dots, F_n$  are pairwise disjoint sets of size  $n$  belonging to  $\mathcal{M} \cap \mathcal{N}$ , then there exists a rainbow set of size  $n - 1$  belonging to  $\mathcal{M} \cap \mathcal{N}$ . Following an idea of Woolbright and Brower-de Vries-Wieringa, we prove that there exists such a rainbow set of size at least  $n - \sqrt{n}$ .

## 1. INTRODUCTION

As in the abstract, a *partial rainbow function* of a family of sets  $\mathcal{F} = (F_1, \dots, F_n)$  is a partial choice function. A *partial rainbow set* is the range of a rainbow function, so it is a set consisting of at most one element from each  $F_i$ , where repeated elements are considered distinct (so, in this terminology a rainbow set is in fact a multiset). A full rainbow set, in which elements are chosen from all  $F_i$ , is called plainly a *rainbow set*. Strengthening a conjecture of Brualdi and Stein [4, 16], Aharoni and Berger [2] made the following conjecture:

**Conjecture 1.1.**  *$n$  matchings of size  $n + 1$  in a bipartite graph have a rainbow matching (namely, a rainbow set that is a matching).*

This conjecture easily implies:

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**Conjecture 1.2.**  *$n$  matchings of size  $n$  in a bipartite graph have a partial rainbow matching of size  $n - 1$ .*

The Brualdi-Stein conjecture is that every Latin square of order  $n$  possesses a partial transversal of size  $n - 1$ , namely  $n - 1$  entries lying in different rows and columns, and containing different symbols. (This is a natural variation on a conjecture of Ryser [14], that an odd Latin square has a full transversal). Each of the  $n$  rows of a Latin square can be considered in a natural way as a matching of size  $n$  between columns and symbols, and applying Conjecture 1.2 to these matchings yields the Brualdi-Stein conjecture.

Lower bounds of order  $n - o(n)$  were proved in both problems. Hatami and Shor [8] proved that in a Latin square of order  $n$  there exists a partial transversal of size at least  $n - 11.053 \log^2 n$ . Woolbright [21] and independently Brouwer, de Vries and Wieringa [3] proved (in effect) that  $n$  matchings in a bipartite graph have a partial rainbow matching of size at least  $n - \sqrt{n}$ .

Aharoni and Berger [2] extended Conjecture 1.2 to matroids, as follows:

**Conjecture 1.3.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two matroids on the same vertex set. Any  $n$  pairwise disjoint sets of size  $n$ , belonging to  $\mathcal{M} \cap \mathcal{N}$ , have a partial rainbow set of size  $n - 1$  belonging to  $\mathcal{M} \cap \mathcal{N}$ .*

Conjecture 1.2 is the special case where both  $\mathcal{M}$  and  $\mathcal{N}$  are partition matroids. (Here the term *partition matroid* refers to a direct sum of uniform matroids, each of rank 1.) The aim of this paper is to prove the parallel of the Woolbright-Brouwer-de Vries-Wieringa result for Conjecture 1.2. We shall prove:

**Theorem 1.4.** *Any  $n$  pairwise disjoint sets of size  $n$  belonging to  $\mathcal{M} \cap \mathcal{N}$  have a partial rainbow set of size at least  $n - \sqrt{n}$  belonging to  $\mathcal{M} \cap \mathcal{N}$ .*

## 2. MATROID PRELIMINARIES

Throughout the paper we shall use the notation  $A + x$  for  $A \cup \{x\}$  and  $A - x$  for  $A \setminus \{x\}$ .

Recall that a collection  $\mathcal{M}$  of subsets of a set  $S$  is a *matroid* if it is hereditary and it satisfies an augmentation property: If  $A, B \in \mathcal{M}$  and  $|B| > |A|$ , then there exists  $x \in B \setminus A$  such that  $A + x \in \mathcal{M}$ . Sets in  $\mathcal{M}$  are called *independent* and sets not belonging to  $\mathcal{M}$  are called *dependent*. A maximal independent set is called a *basis*. An element  $x \in S$  is *spanned* by  $A$  if either  $x \in A$  or  $I + x \notin \mathcal{M}$  for some independent set  $I \subseteq A$ . The set of elements that are spanned by  $A$  is denoted by  $\text{sp}(A)$ , or  $\text{sp}_{\mathcal{M}}(A)$  if the identity of the matroid  $\mathcal{M}$  is not clear from the context. A *circuit* is a minimal dependent set. We shall use some basic facts on matroids, that can be found, for example, in the books of Oxley [13] and Welsh [20].

**Fact 2.1.** *If  $I$  is independent and  $I + x$  is dependent, then there exists a unique minimal subset  $C_{\mathcal{M}}(I, x)$  of  $I$  that spans  $x$ .*

We shall call  $C_{\mathcal{M}}(I, x)$  the  $\mathcal{M}$ -support of  $x$  in  $I$ .

**Fact 2.2.** *Let  $A \in \mathcal{M}$ ,  $x \in \text{sp}(A)$ , and  $a \in C_{\mathcal{M}}(A, x)$ . Then  $A + x - a \in \mathcal{M}$  and  $\text{sp}(A + x - a) = \text{sp}(A)$ .*

**Fact 2.3.** *If  $C_1$  and  $C_2$  are circuits with  $e \in C_1 \cap C_2$  and  $f \in C_1 \setminus C_2$  then there exists a circuit  $C_3$  such that  $f \in C_3 \subseteq (C_1 \cup C_2) - e$ .*

The following is an immediate corollary of the augmentation property:

**Fact 2.4.** *Let  $I, J$  be independent sets in  $\mathcal{M}$ . If  $|I| < |J|$ , then there exists  $J_1 \subseteq J \setminus I$  such that  $I \cup J_1 \in \mathcal{M}$  and  $|I \cup J_1| = |J|$ .*

**Definition 2.5.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be two matroids on the same ground set  $S$ . We call a set  $F \subseteq S$  an *independent matching* if  $F \in \mathcal{M} \cap \mathcal{N}$ . A rainbow set for a family  $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$  of independent matchings is called a *rainbow independent matching* if it belongs to  $\mathcal{M} \cap \mathcal{N}$ .

### 3. PROOF OF THEOREM 1.4

Let  $\mathcal{F} = (F_1, \dots, F_n)$  be a family of disjoint sets belonging to  $\mathcal{M} \cap \mathcal{N}$ . Let  $R$  be a partial rainbow matching for  $\mathcal{F}$  of maximal size. Let  $t = |R|$  and  $\delta = n - t$ . Without loss of generality we may assume that  $|R \cap F_i| = 1$  for  $i = 1, \dots, t$ .

We shall construct a sequence of sets  $(A_1, \dots, A_\delta)$  such that for all  $i = 1, \dots, \delta$  the following holds:

$$(3.1) \quad A_i \subseteq F_{t+i},$$

$$(3.2) \quad A_i \subseteq \text{sp}_{\mathcal{M}}(R),$$

and

$$(3.3) \quad |A_i| \geq i\delta.$$

Suppose that we succeed in constructing such a sequence. By (3.1)  $A_\delta \in \mathcal{M}$  and by (3.2)  $A_\delta \subseteq \text{sp}_{\mathcal{M}}(R)$ . By (3.3), applied to  $i = \delta$ , we therefore have  $t = |R| \geq |A_\delta| \geq \delta^2$ . Clearly, we may assume that  $t < n$ . Since  $\delta = n - t$ , it follows that  $t > n - \sqrt{n}$ , as stated in the theorem.

*Construction of the sets  $A_i$ .* We construct the sets  $A_i$  inductively, associating with them sets  $R_i$ , so that  $R_1, \dots, R_\delta$  are disjoint,  $R_i \subseteq R$  and  $|R_i| \geq \delta$  for all  $i = 1, \dots, \delta$ . Since  $|F_{t+1}| = n$  and  $|R| = t$ , there exists, by Fact 2.4, a set  $A_1 \subseteq F_{t+1} \setminus R$  such that  $|A_1| = \delta$  and  $R \cup A_1 \in \mathcal{N}$ . By the maximality property of  $R$  we have  $A_1 \subseteq \text{sp}_{\mathcal{M}}(R)$ . Since  $|A_1| = \delta$  and  $|R| = t$ , there exists, again by Fact 2.4, a subset  $R' \subset R$  of size  $t - \delta$  such that  $A_1 \cup R' \in \mathcal{M}$  and  $|A_1 \cup R'| = t$ . Let  $R_1 = R \setminus R'$ . We have  $R \setminus R_1 \cup A_1 \in \mathcal{M}$  and  $|R_1| = \delta$ .

For the inductive step, assume that  $R_1, R_2, \dots, R_j$  are pairwise disjoint subsets of  $R$ , each of size at least  $\delta$ , and  $A_1, A_2, \dots, A_j$  satisfy the conditions (3.1), (3.2) and (3.3), for  $i = 1, \dots, j$ . Denote  $R^k = R \setminus \bigcup_{i=1}^{k-1} R_i$  for  $k = 2, \dots$ . Notice that  $|R^{j+1}| \leq t - j\delta$ . Since  $F_{t+j+1} \in \mathcal{N}$  and  $|F_{t+j+1}| = n$  it follows from Fact 2.4 that there exists  $A_{j+1} \subseteq F_{t+j+1}$  such that  $R^{j+1} \cup A_{j+1} \in \mathcal{N}$  and  $|R^{j+1} \cup A_{j+1}| = n$ . We have  $|A_{j+1}| = n - |R^{j+1}| \geq n - (t - j\delta) = (j+1)\delta$ . We see that  $A_{j+1}$  satisfies (3.1) and (3.3). The following lemma implies that (3.2) also holds for  $A_{j+1}$ .

**Lemma 3.1.** *If  $j < \delta$  then  $A_{j+1} \subseteq \text{sp}_{\mathcal{M}}(R)$ .*

Before proving Lemma 3.1 let us indicate how it is used to complete the inductive construction of  $R_{j+1}$ . We use the following observation:

**Observation 3.2.** *Let  $I$  be an independent set of size  $t$  in a matroid  $\mathcal{M}$  and suppose  $J \subseteq \text{sp}(I)$  has size  $n > t$ . If  $K \subset J$  satisfies  $J \setminus K \in \mathcal{M}$ , then  $|K| \geq n - t$ .*

Assuming Lemma 3.1, we have (\*)  $A_{j+1} \subseteq \text{sp}_{\mathcal{M}}(R)$ . We also have  $|R^{j+1} \cup A_{j+1}| = n = |R| + \delta$ . Hence  $|R^{j+1}| \geq \delta$  (If  $|R^{j+1}| < \delta$  then  $|A_{j+1}| > |R|$ , contradicting (\*)). Let  $R_{j+1} \subset R^{j+1}$  be of minimal size such that  $R^{j+1} \setminus R_{j+1} \cup A_{j+1} \in \mathcal{M}$ . By Observation 3.2 we have  $|R_{j+1}| \geq \delta$ , as required.

The proof of Lemma 3.1 is done by an alternating path argument.

**Definition 3.3.** A *colorful alternating path* (CAP) of length  $k$ , relative to  $R$ , consists of

- (i) A set  $\{b_0, b_1, \dots, b_k\}$  of distinct elements of the ground set  $S$ , where each  $b_i$  belongs to some  $A_j \in \mathcal{A}$  and distinct  $b_i$ 's belong to distinct  $A_j$ 's.
- (ii) A set of distinct elements  $\{r_1, \dots, r_k\} \subseteq R$  such that
  - (P $_{\mathcal{M}}$ )  $R - r_1 + b_1 - r_2 + b_2 - \dots - r_k + b_k \in \mathcal{M}$  and  $\text{sp}_{\mathcal{M}}(R - r_1 + b_1 - r_2 + b_2 - \dots - r_k + b_k) = \text{sp}_{\mathcal{M}}(R)$ .
  - (P $_{\mathcal{N}}$ )  $R + b_0 - r_1 + b_1 - r_2 + \dots + b_{k-1} - r_k \in \mathcal{N}$  and  $\text{sp}_{\mathcal{N}}(R + b_0 - r_1 + b_1 - r_2 + \dots + b_{k-1} - r_k) = \text{sp}_{\mathcal{N}}(R)$ .

If, in addition,  $R + b_0 - r_1 + b_1 - r_2 + b_2 - \dots - r_k + b_k \in \mathcal{M} \cap \mathcal{N}$  then the CAP is called *augmenting*.

Since the  $b_i$ 's belong to distinct  $F_{t+j}$ 's we have:

**Observation 3.4.** *If  $R$  is of maximal size then no augmenting CAP relative to  $R$  exists.*

In order to extend our alternating path we shall need the following lemma:

**Lemma 3.5.** *Let  $\mathcal{M}$  be a matroid. Let  $I \in \mathcal{M}$  and  $X = \{x_1, \dots, x_k\} \subseteq I$  and  $Y = \{y_1, \dots, y_k\} \subseteq \text{sp}_{\mathcal{M}}(I) \setminus I$  be such that  $\text{sp}_{\mathcal{M}}((I \setminus X) \cup Y) = \text{sp}_{\mathcal{M}}(I)$ . Suppose  $y_{k+1} \in \text{sp}_{\mathcal{M}}(I) \setminus I$  and  $x_{k+1}$  are such that  $x_{k+1} \in C_{\mathcal{M}}(I, y_{k+1}) \setminus X$  and  $x_{k+1} \notin C_{\mathcal{M}}(I, y_i)$  for all  $i = 1, \dots, k$ . Then  $x_{k+1} \in C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1})$ .*

*Proof of Lemma 3.5.* Suppose, for contradiction, that  $x_{k+1} \notin C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1})$ . Let  $C' = C_{\mathcal{M}}(I, y_{k+1}) + y_{k+1}$  and  $C'' = C_{\mathcal{M}}((I \setminus X) \cup Y, y_{k+1}) + y_{k+1}$ . Then, by Fact 2.3, there exists a circuit  $C \subseteq C' \cup C''$ , such that  $x_{k+1} \in C$  and  $y_{k+1} \notin C$ . Choose such a circuit  $C$  with  $|C \cap Y|$  minimal. Since  $I$  is independent  $C$  must contain at least one element  $y_j \in Y \cap C''$ . Using Fact 2.3 again, since  $x_{k+1} \notin C_{\mathcal{M}}(I, y_j)$ , there exists a circuit  $\tilde{C} \subseteq C \cup (C_{\mathcal{M}}(I, y_j) + y_j)$  such that  $x_{k+1} \in \tilde{C}$  and  $y_j \notin \tilde{C}$ . We have  $|\tilde{C} \cap Y| < |C \cap Y|$ , contradicting the minimality property of  $C$ .  $\square$

*Proof of Lemma 3.1.* We shall show how the existence of some  $i$ ,  $1 \leq i \leq \delta$ , such that  $A_i \not\subseteq \text{sp}_{\mathcal{M}}(R)$  yields an augmenting CAP relative to  $R$ . This will contradict the maximality of  $R$ , by Observation 3.4.

Let  $\{A_i\}$ ,  $\{R_i\}$  and  $\{R^i\}$  be defined as above. Recall that for all  $i = 1, \dots, \delta$ ,

$$(3.4) \quad R^i = R \setminus \bigcup_{j=1}^{i-1} R_j,$$

$$(3.5) \quad A_i \subseteq F_{t+i} \text{ satisfies } R^i \cup A_i \in \mathcal{N} \text{ and } |R^i \cup A_i| = n$$

and

$$(3.6) \quad R_i \subseteq R^i \text{ is of minimal size such that } R^i \setminus R_i \cup A_i \in \mathcal{M}.$$

Assume, for contradiction, that  $m$ ,  $1 \leq m \leq \delta$ , is the minimal index such that  $A_m \not\subseteq \text{sp}_{\mathcal{M}}(R)$  and let  $a \in A_m$  be such that  $R+a \in \mathcal{M}$ . We shall construct a CAP, relative to  $R$ , starting from  $a$ . Let  $b_0 = a$ . We have

$$(3.7) \quad R + b_0 \in \mathcal{M}$$

and, since no augmenting CAP relative to  $R$  exists, we must have  $b_0 \in \text{sp}_{\mathcal{N}}(R)$ . Let  $j$  be the maximal index such that  $b_0 \in \text{sp}_{\mathcal{N}}(R^j)$ . Since  $b_0 \in A_m$  and, by (3.5),  $R^m \cup A_m \in \mathcal{N}$ , we obtain  $b_0 \notin \text{sp}_{\mathcal{N}}(R^m)$ . Thus,  $j < m$ . Since  $R_j = R^j \setminus R^{j+1}$ , it follows from the maximality of  $j$  that  $C_{\mathcal{N}}(R^j, b_0) \cap R_j \neq \emptyset$ . By Fact 2.2, there exists  $r_1 \in R_j$  such that  $R + b_0 - r_1 \in \mathcal{N}$  and

$$(3.8) \quad \text{sp}_{\mathcal{N}}(R + b_0 - r_1) = \text{sp}_{\mathcal{N}}(R).$$

Since  $j < m$ , we have, by the minimality of  $m$ , that  $A_j \subseteq \text{sp}_{\mathcal{M}}(R)$ . By the minimality of  $R_j$  (see (3.6)) there exists  $x \in A_j$  such that  $r_1 \in C_{\mathcal{M}}(R, x)$  (otherwise  $A_j \cup R^{j+1} + r_1 \in \mathcal{M}$ ). Let  $l \leq j$  be minimal such that  $A_l$  contains an element  $b_1$  satisfying  $r_1 \in C_{\mathcal{M}}(R, b_1)$ . By Fact 2.2, we have  $R - r_1 + b_1 \in \mathcal{M}$  and  $\text{sp}_{\mathcal{M}}(R - r_1 + b_1) = \text{sp}_{\mathcal{M}}(R)$ . This, combined with (3.7), implies that  $R + b_0 - r_1 + b_1 \in \mathcal{M}$ . Thus, a CAP of length 1 was created.

Now, suppose that we managed to construct a CAP of length  $k$ . We shall show that if the CAP is not augmenting, then it can be extended. Denote  $R_{\mathcal{M}}(k) = R - r_1 + b_1 - r_2 + b_2 - \dots - r_k + b_k$  and  $R_{\mathcal{N}}(k) = R + b_0 - r_1 + b_1 - r_2 + \dots + b_{k-1} - r_k$ . Note that

$$(3.9) \quad R_{\mathcal{M}}(k) + b_0 = R_{\mathcal{N}}(k) + b_k.$$

*Claim 1.*  $b_k \in \text{sp}_{\mathcal{N}}(R)$ .

*Proof of Claim 1.* By  $(P_{\mathcal{M}})$ , we have  $\text{sp}_{\mathcal{M}}(R_{\mathcal{M}}(k)) = \text{sp}_{\mathcal{M}}(R)$ . Hence, from (3.7) we have  $R_{\mathcal{M}}(k) + b_0 \in \mathcal{M}$ . Also, by  $(P_{\mathcal{N}})$ , we have  $\text{sp}_{\mathcal{N}}(R_{\mathcal{N}}(k)) = \text{sp}_{\mathcal{N}}(R)$ . Assume, for contradiction, that  $R + b_k \in \mathcal{N}$ . Then,  $R_{\mathcal{N}}(k) + b_k \in \mathcal{N}$ , and by (3.9) we obtain an augmenting CAP, contradicting the maximality property of  $R$ .

Assuming Claim 1, let  $p$  be the maximal index such that  $b_k \in \text{sp}_{\mathcal{N}}(R^p)$ . By (3.4),  $p$  is the minimal index such that  $C_{\mathcal{N}}(R, b_k) \cap R_p \neq \emptyset$ . Let  $r_{k+1} \in C_{\mathcal{N}}(R, b_k) \cap R_p$ . By Fact 2.2,  $R + b_k - r_{k+1} \in \mathcal{N}$  and  $\text{sp}_{\mathcal{N}}(R + b_k - r_{k+1}) = \text{sp}_{\mathcal{N}}(R)$ .

*Claim 2.* Let  $q$  be the index such that  $b_k \in A_q$ . Then,  $p < q$ .

*Proof of Claim 2.* By (3.5),  $R^q \cup A_q \in \mathcal{N}$  and thus,  $b_k \notin \text{sp}_{\mathcal{N}}(R^q)$ . Since  $b_k \in \text{sp}_{\mathcal{N}}(R^p)$  we conclude that  $R^q \subsetneq R^p$ , which implies that  $p < q$ .

*Claim 3.* There exists  $x \in A_p$  such that  $r_{k+1} \in C_{\mathcal{M}}(R, x)$ .

*Proof of Claim 3.* Assume the opposite. Then  $A_p \cup R^{p+1} + r_{k+1} \in \mathcal{M}$ . This contradicts the minimality property of  $R_p$  (see (3.6)).

Let  $l$  be minimal such that  $A_l$  contains an element  $b_{k+1}$  satisfying  $r_{k+1} \in C_{\mathcal{M}}(R, b_{k+1})$ . By Claim 3,  $l \leq p$ . This, together with Claim 2, yields

$$(3.10) \quad \text{if } b_i \in A_u \text{ and } b_j \in A_v \text{ with } i < j, \text{ then } v < u,$$

and

$$(3.11) \quad \text{if } r_i \in R_u \text{ and } r_j \in R_v \text{ with } i < j, \text{ then } v < u.$$

*Claim 4.*  $r_{k+1} \notin C_{\mathcal{N}}(R, b_i)$  for all  $i = 0, \dots, k-1$ .

*Proof of Claim 4.* Let  $j \in \{1, \dots, k\}$ . In the construction described above, the element  $r_j$  was chosen from  $R_u$ , where  $u$  is minimal such that  $C_{\mathcal{N}}(R, b_{j-1}) \cap R_u \neq \emptyset$ . Recall that  $r_{k+1} \in R_p$ . Thus, by (3.11), we have  $p < u$ , and hence  $C_{\mathcal{N}}(R, b_{j-1}) \cap R_p = \emptyset$ , which implies the claim.

By applying Lemma 3.5 to the sequences  $\{r_1, \dots, r_k, r_{k+1}\}$  and  $\{b_0, \dots, b_{k-1}, b_k\}$ , it follows that  $r_{k+1} \in C_{\mathcal{N}}(R_{\mathcal{N}}(k), b_k)$ . By Fact 2.2, it follows that

$$(3.12) \quad \begin{aligned} &R_{\mathcal{N}}(k) + b_k - r_{k+1} \in \mathcal{N}, \text{ and} \\ &\text{sp}_{\mathcal{N}}(R_{\mathcal{N}}(k) + b_k - r_{k+1}) = \text{sp}_{\mathcal{N}}(R_{\mathcal{N}}(k)) = \text{sp}_{\mathcal{N}}(R). \end{aligned}$$

*Claim 5.*  $r_{k+1} \in C_{\mathcal{M}}(R_{\mathcal{M}}(k), b_{k+1})$ .

*Proof of Claim 5.* Let  $i \in \{1, \dots, k\}$ . In the construction described above, the element  $b_i$  was chosen from  $R_u$ , where  $u$  is minimal such that  $A_u$  contains an element  $b_i$  such that  $r_i \in C_{\mathcal{M}}(R, b_i)$ . Recall that  $b_{k+1}$  was chosen from  $A_l$ , and by (3.10),  $l < u$ . Thus,  $r_i \notin C_{\mathcal{M}}(R, b_{k+1})$ . Since this is true for any  $i \in \{1, \dots, k\}$ , we have  $C_{\mathcal{M}}(R, b_{k+1}) \cap \{r_1, \dots, r_k\} = \emptyset$ , and hence,  $C_{\mathcal{M}}(R_{\mathcal{M}}(k), b_{k+1}) = C_{\mathcal{M}}(R, b_{k+1})$ . Since  $b_{k+1}$  was chosen so that  $r_{k+1} \in C_{\mathcal{M}}(R, b_{k+1})$ , the claim follows.

Assuming Claim 5, by Fact 2.2, we have

$$(3.13) \quad \begin{aligned} &R_{\mathcal{M}}(k) + b_{k+1} - r_{k+1} \in \mathcal{M}, \text{ and} \\ &\text{sp}_{\mathcal{M}}(R_{\mathcal{M}}(k) + b_{k+1} - r_{k+1}) = \text{sp}_{\mathcal{M}}(R_{\mathcal{M}}(k)) = \text{sp}_{\mathcal{M}}(R). \end{aligned}$$

By  $(P_{\mathcal{M}})$ ,  $(P_{\mathcal{N}})$ , (3.12) and (3.13), the CAP was extended to the length of  $k+1$ .

By (3.10) and (3.11), the process must end, yielding an augmenting CAP. This completes the proof of Lemma 3.1 and hence of Theorem 1.4.  $\square$

#### 4. INDEPENDENT PARTIAL TRANSVERSALS IN MATROIDAL LATIN SQUARES

Let  $\mathcal{M}$  be matroid of rank  $n$  defined on a ground set  $S$ . A *Matroidal Latin Square (MLS)* of degree  $n$  over  $\mathcal{M}$  was defined in [10] as an  $n \times n$  matrix whose entries are from  $S$ , such that each row and column is a basis of  $\mathcal{M}$ . (After publication, the authors found out that a similar object had been introduced earlier by Chappell [5].) Note that the notion of MLS generalizes the notion of Latin square, as a Latin square is an MLS over a partition matroid (that is, a direct sum of uniform matroids, each of rank 1). Analogously to Norton's definition for row Latin square

in [12], we define a *row MLS*, as an  $n \times n$  matrix whose entries are from  $S$ , such that each row is a basis of  $\mathcal{M}$ . Thus, every MLS is a row MLS.

An *independent partial transversal* in an MLS, or in a row MLS,  $A$ , is an independent subset of entries of  $A$  where no two of them lie in the same row or column of  $A$ . It was conjectured in [10] that every MLS of degree  $n$  has an independent partial transversal of size  $n - 1$ . It was shown there that, in general, we cannot expect to find a partial independent transversal of size  $n$ . The lower bound set in [10] for the size of a partial independent transversal in an MLS was  $\lceil 2n/3 \rceil$ . Theorem 1.4 yields a significant improvement for that bound:

**Corollary 4.1.** *Every row MLS of degree  $n$  has an independent partial transversal of size at least  $n - \sqrt{n}$ .*

*Proof.* Let  $A$  be a row MLS of degree  $n$  over a matroid  $\mathcal{M}$ . The result follows from Theorem 1.4 by taking  $\mathcal{N}$  as the partition matroid defined by the columns of  $A$ .  $\square$

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